The Natural Numbers

$\mathbb{N} = 1, 2, 3, 4, 5, \ldots$

i.e. the counting numbers.

We won't provide a construction for $\mathbb{N}$ but rather regard $\mathbb{N}$ as too intuitive for such to be necessary.

$\mathbb{N}$ comes equipped with addition and multiplication such that

Properties of Addition + Multiplication

Let $m, n, k \in \mathbb{N}$.

1) $m+n=n+m$ (commutativity of $+$)

2) $m \cdot n = n \cdot m$ (commutativity of $\cdot$)

3) $m+(n+k)=(m+n)+k$ (associativity of $+$)

4) $m \cdot (n \cdot k) = (m \cdot n) \cdot k$ (associativity of $\cdot$)

5) $m \cdot (n+k) = m \cdot n + m \cdot k$

(Distributivity of $\cdot$ over $+$)

If $m > n$, define $m - n$ to be the element $k \in \mathbb{N}$ such that $n + k = m$.

The Grothendieck Group Construction

We're lacking negative numbers and zero. We want to construct the integers:

$$\mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$$

Consider

$$\mathbb{N} \times \mathbb{N}$$

and define addition and multiplication coordinate-wise:

$$(n_1, n_2) + (m_1, m_2)$$

$$= (n_1 + m_1, n_2 + m_2)$$

$$(n_1, n_2) \cdot (m_1, m_2)$$

$$= (n_1 m_1, n_2 m_2)$$

Now define an equivalence relation on $\mathbb{N} \times \mathbb{N}$ by

$$(n_1, n_2) \sim (m_1, m_2)$$

if $\boxed{n_1 + m_2 = n_2 + m_1}$

Define $\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim$

Now we just need to check that this is an equivalence relation!

Let $(m_1, m_2), (n_1, n_2),$

$(k_1, k_2) \in \mathbb{N} \times \mathbb{N}$

1) $\underline{(m_1, m_2) \sim (m_1, m_2)}$

Since $n_1 = m_1$, $n_2 = m_2$

$$\begin{aligned} n_1 + m_2 &= m_1 + m_2 \\ &= m_1 + n_2 \\ &= n_2 + m_1 \quad \checkmark \end{aligned}$$

2) If $(n_1, n_2) \sim (m_1, m_2)$, then $(m_1, m_2) \sim (n_1, n_2)$

Let $(n_1, n_2) \sim (m_1, m_2)$.

Then $n_1 + m_2 = n_2 + m_1$.

By commutativity of addition,

$$m_2 + n_1 = m_1 + n_2$$

$\Rightarrow (m_1, m_2) \sim (n_1, n_2)$ ✓

3) If $(m_1, m_2) \sim (n_1, n_2)$ and $(n_1, n_2) \sim (k_1, k_2)$, then $(m_1, m_2) \sim (k_1, k_2)$.

If $(m_1, m_2) \sim (n_1, n_2)$, then

$$m_1 + n_2 = m_2 + n.$$

If $(n_1, n_2) \sim (k_1, k_2)$, then

$$n_1 + k_2 = n_2 + k_1.$$

Solving for n_2, $n_2 = n_1 + k_2 - k_1$.

Substituting into

$$m_1 + n_2 = m_2 + n_1,$$

$$m_1 + (n_1 + k_2 - k_1) = m_2 + n_1$$

$$m_1 + n_1 + k_2 - k_1 = m_2 + n_1$$

$$-n_1 \quad +k_1 \qquad\qquad -n_1 + k_1$$

$$m_1 + k_2 = m_2 + k_1$$

$$\Rightarrow (m_1, m_2) \sim (k_1, k_2) \checkmark$$

So we know $\sim$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

Now we need to see why

$\mathbb{N} \times \mathbb{N} / \sim$ should be $\mathbb{Z}$.

Define, for an equivalence relation $\sim$ on a set A and $a \in A$,

$$\boxed{[a] = \{ b \in A \mid b \sim a \}},$$

the equivalence class of a.

For $[(m_1,m_2)]$, $[(n_1,n_2)]$

define $\boxed{[(m_1,m_2)]+[(n_1,n_2)] = [(m_1+n_1,\ m_2+n_2)].}$

Let us check this is well-defined.

That is, if

$$(k_1,k_2)\sim(m_1,m_2)$$

and $(\ell_1,\ell_2)\sim(n_1,n_2)$, then

$$(m_1+n_1,m_2+n_2)\sim(k_1+\ell_1,k_2+\ell_2)$$

Since $(m_1, m_2) \sim (k_1, k_2)$
and $(n_1, n_2) \sim (l_1, l_2)$,

$$m_1 + k_2 = m_2 + k_1 \text{ and}$$
$$n_1 + l_2 = n_2 + l_1.$$

Adding these equations,

$$(m_1 + k_2) + (n_1 + l_2) = (m_2 + k_1) + (n_2 + l_1)$$

Happily rearranging parentheses and using commutativity,

$$(m_1 + n_1) + (k_2 + l_2) = (m_2 + n_2) + (k_1 + l_1).$$
$$\Rightarrow (m_1 + n_1, m_2 + n_2) \sim (k_1 + l_1, k_2 + l_2) \checkmark$$

Hence, addition is well-defined.

Claim: $0 = [(1,1)]$.

Let $(m_1, m_2) \in \mathbb{N} \times \mathbb{N}$.

$$[(m_1, m_2)] + [(1,1)]$$

$$= [(\underbrace{m_1 + 1}_{n_1}, \underbrace{m_2 + 1}_{n_2})]$$

Then $m_1 + n_2 = m_1 + m_2 + 1$

$$= m_2 + m_1 + 1$$

$$= m_2 + n_1 \checkmark$$

Then $[(1,1)] + [(m_1, m_2)]$

$= [(m_1, m_2)]$

$\forall\ (m_1, m_2) \in \mathbb{N} \times \mathbb{N}$.

So now we know how to define zero. What counts as a positive or negative number?

The idea: the pair (m_1, m_2) is supposed to correspond to the integer

$$m_1 - m_2$$

So $[(1,1)] = 1 - 1 = 0$

We want to set positive numbers to be the equivalence class of

$$[(m+1, 1)] = m$$

For negative numbers:

$$[(1, m+1)] = -m$$

Observation:

Take $[(m_1, m_2)]$ with

$m_1 > m_2$.

Claim:

$$(m_1, m_2) \sim (m_1 - m_2 + 1, 1)$$

check: $m_1 + 1$

$$= m_2 + (m_1 - m_2 + 1)$$

$$= m_1 + 1 \checkmark$$

If $m_2 > m_1$, then

I claim

$$(m_1, m_2)$$

$$\sim (1, m_2 - m_1 + 1).$$

Check:

$$\underbrace{m_1 + (m_2 - m_1 + 1)}_{m_2 + 1} = m_2 + 1$$

$m_2 + 1$ ✓

Multiplication

If (m_1, m_2) represents $m_1 - m_2$ and (n_1, n_2) represents $n_1 - n_2$, then

$(m_1, m_2)(n_1, n_2)$

should equal

$$(m_1 - m_2)(n_1 - n_2)$$

$$= \underbrace{m_1 n_1 + m_2 n_2}_{\text{positive part}} - \underbrace{m_2 n_1 - m_1 n_2}_{\text{negative part}}$$

Define

$$\boxed{[(m_1, m_2)] \cdot [(n_1, n_2)] = [(m_1 n_1 + m_2 n_2, m_1 n_2 + m_2 n_1)]}$$

Need to check that this is correct, i.e., consistent with our idea of multiplication

This is well-defined, but I won't check it!

Other Properties

1) $(-1)\cdot n = -n \quad \forall\, n \in \mathbb{Z}$

$-1 = [(1,2)]$. Let $m \in \mathbb{N}$.

$$[(1,2)]\cdot[(m+1,1)]$$

$$= [(m+1+2,\ 2(m+1)+1)]$$

$$= [(m+3,\ 2m+3)]$$

$$= [(1,\ m+1)]$$

The last equality is true since

$$(m+3)+(m+1)$$
$$= 2m+4$$
$$= (2m+3)+1$$

A similar calculation shows

$$[(1,2)]\cdot[(1,m+1)]$$
$$= [(m+1,1)].$$

2) $\underline{0 \cdot n = 0 \quad \forall \; n \in \mathbb{Z}}$

Let $[(m_1, m_2)] \in \mathbb{N} \times \mathbb{N} / \sim$

$$[(1,1)] \cdot [(m_1, m_2)]$$

$$= [(m_1 + m_2, m_1 + m_2)]$$

$$= [(1,1)] \checkmark$$

3) Commutativity

Let $m_1, m_2, n_1, n_2 \in \mathbb{N}$.

$$[(m_1, m_2)] \cdot [(n_1, n_2)]$$

$$= [(m_1 n_1 + m_2 n_2, m_1 n_2 + m_2 n_1)]$$

$$= [(n_1 m_1 + n_2 m_2, n_2 m_1 + n_1 m_2)]$$

$$= [(n_1, n_2)] \cdot [(m_1, m_2)] \checkmark$$

4) $\underline{n^2 > 0 \ \forall \ n \neq 0}$

Let $[(m_1, m_2)] \in \mathbb{N} \times \mathbb{N} / \sim$,

$m_1 \neq m_2$.

$$[(m_1, m_2)] \cdot [(m_1, m_2)]$$

$$= [(m_1^2 + m_2^2 , 2m_1 m_2)]$$

Note

$$m_1^2 + m_2^2 - 2m_1 m_2$$

$$= (m_1 + m_2)^2 \in \mathbb{N}$$

This shows

$m_1^2 + m_2^2$ is greater

than $2m_1m_2$.

Of course need to check

many other properties

(associativity,

distributivity, etc.)

but enough!

Ordering

Define an order "$<$" on $\mathbb{N}\times\mathbb{N}/\sim$ by:

let $m \in \mathbb{N}$. Then

$$[(m+1,1)] > [(1,1)]$$

and if $n \in \mathbb{N}$ and $m > n$, then

$$[(m+1,1)] > [(n+1,1)].$$

Set, for $m, n \in \mathbb{N}$, $m > n$,

$$[(1, m+1)] < [(1,1)]$$

$$[(1, m+1)] < [(k+1, 1)]$$

$\forall\ k \in \mathbb{N}$ and

$$[(1, m+1)] < [(1, n+1)].$$

Subtraction

Observe: let $m, n \in \mathbb{N}$.

$$[(m,n)] + [(n,m)]$$

$$= [(m+n, m+n)]$$

$$= [(1,1)]$$

So every $[(m,n)] \in \mathbb{N}\times\mathbb{N}/\sim$

has additive inverse, i.e.,

an element $[(k,l)]$ with

$$[(m,n)] + [(k,\ell)]$$
$$= [(1,1)].$$

Define $[(m,n)] - [(k,\ell)]$
$$:= [(m,n)] + [(\ell,k)]$$

Hopefully we are now convinced that
$$\mathbb{N}\times\mathbb{N}/\sim \;=\; \mathbb{Z}.$$